right member and trouble the portion $m''R_1$. Thus we have the strange spectacle of forces figuring at once as principal and as disturbing. Mr. Stockwell made a precisely similar objection to my elaboration of the inequalities due to the figure of the Earth, which was disposed of by Prof. Adams in a single sentence.

If all this be true, what becomes of the assertion, often reiterated, that, when the differential equations are written down, all the rest is a pure question of analysis? On Mr. Neison's and Mr. Stockwell's view, the analyst, who does the integrating, needs an astronomical or mechanical prompter at his elbow to inform him of the exact physical import of the constants $\beta$ or $m''$, otherwise he will infallibly go wrong.

*Washington,*
1886, *Oct.* 14.

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## *On Kepler's Problem.* By Robert Bryant, B.A., B.Sc.

The solutions of this problem are usually given with a view of obtaining the eccentric anomaly directly from a series expanded in powers of the eccentricity, the application of which is in general very laborious.

The use of the differential formula

$$\frac{dE}{dM}=\frac{1}{1-e\cos E},$$

when an approximate value of E in a planetary orbit is known, involves less labour, while the results converge with great rapidity.

The two following series are given as readily offering the required approximate value of E in a planetary orbit, when one application of the differential formula usually gives the eccentric anomaly with sufficient accuracy. The series are so simple and so readily present themselves that they have probably been given before, but I have not met with them.

We have $M=E-e\sin E$ with M and $e$, given to find E.

Lagrange's theorem may be concisely expressed thus:

If

$$y=z+x\phi(y),\text{ then } f(y)=\sum_{n=0}^{n=\infty}\frac{x^n}{\underline{|n}}\frac{d^{n-1}}{dz^{n-1}}\left\{\overline{\phi(z)}|^n f'(z)\right\},$$

the interpretation of which when $n=0$ is obvious.

If we expand sin E and cos E, we obtain

$$\sin E=\sin M+e\sin M\cos M+\frac{e^2}{2}\sin M(2-3\sin^2 M)+\frac{e^3}{3}\sin M\cos M(3-8\sin^2 M)$$
$$+\frac{e^4}{24}\sin M(24-136\sin M+125\sin^4 M)+\ldots.$$

$$\cos E=\cos M-e\sin^2 M-\frac{3e^2}{2}\sin^2 M\cos M-\frac{2e^3}{3}\sin^2 M(3-4\sin^2 M)$$
$$+\frac{5e^4}{24}\sin^2 M\cos M(12-4\sin^2 M)+\ldots.$$

These series are generally unsuited for the evaluation of E; but if we multiply them by sin M and cos M respectively, we get on addition

$$\cos(\mathrm{E}-\mathrm{M}) = 1 - \frac{e^2}{2}\sin^2\mathrm{M} - e^3\sin^2\mathrm{M}\cos\mathrm{M} - \frac{e^4}{24}\sin^2\mathrm{M}\,(36-49\sin^2\mathrm{M}) \quad . \; . \; . \; .$$

or

$$\sin^2\tfrac{1}{2}(\mathrm{E}-\mathrm{M}) = \frac{e^2\sin^2\mathrm{M}}{4}(1+2e\cos\mathrm{M}) + \frac{e^4}{24}\sin^2\mathrm{M}\,(36-49\sin^2\mathrm{M}) \qquad \text{(A)}$$

$$= \frac{e^2\sin^2\mathrm{M}}{4}(1+2e\cos\mathrm{M}) + \frac{e^4}{2}\sin\mathrm{M}\sin 3\mathrm{M} \text{ nearly} \qquad \text{(B)}$$

If, as Mr. Glaisher has suggested to me, we extract the square root of each side of (A), we get

$$\sin\tfrac{1}{2}(\mathrm{E}-\mathrm{M}) = \tfrac{1}{2}e\sin\mathrm{M}\left\{1+e\cos\mathrm{M}+\frac{e^2}{12}(30-43\sin^2\mathrm{M})\right\} \text{nearly}$$

$$= \tfrac{1}{2}e\sin\mathrm{M}\left\{1+e\cos\mathrm{M}+\frac{5e^2}{6}\,\frac{\sin 3\mathrm{M}}{\sin\mathrm{M}}\right\} \text{nearly} \qquad \text{(C)}$$

In (B) and (C) the term involving sin 3 M may be neglected in obtaining an approximate value of E.

Again, in the above expression for Lagrange's theorem, if for a particular value of $n$ we can make the quantity within the brackets constant, the corresponding term in $f(y)$ will vanish.

As the form of the function $f$ is completely at our disposal, we may for a particular value of $n$ put

$$\overline{\phi(z)}|^n f'(z) = \text{const.}$$

or

$$f(z) = a\int\frac{dz}{\overline{\phi(z)}|^n}$$

In the present case $\phi(\mathrm{M}) = \sin \mathrm{M}$.

If $n=1$, we derive but little assistance.

If $n=2$, we get $f(\mathrm{M}) = \cot \mathrm{M}$ (omitting a constant multiplier), and $e^2$ will not appear in our result.

If $n=3$,

$$f(\mathrm{M}) = -\frac{\cos\mathrm{M}}{2\sin^2\mathrm{M}} + \tfrac{1}{2}\log\tan\frac{\mathrm{M}}{2},$$

which is too complicated to assist us.

For values of $n$ greater than 3 $f(\mathrm{M})$ is equally unsuitable.

Expanding, then, cot E, we get

$$\cot\mathrm{E} = \cot\mathrm{M} - \frac{e}{\sin\mathrm{M}} + \frac{e^3}{6}\sin\mathrm{M} + \frac{e^4}{3}\sin^2\mathrm{M}$$

$$- \frac{e^5}{40}\sin\mathrm{M}\,(27\sin^2\mathrm{M}-20) \quad . \; . \; . \; .$$

$$= \frac{\cos\mathrm{M}-e}{\sin\mathrm{M}} + \frac{e^3}{6}\sin\mathrm{M} \text{ nearly} \qquad \text{(D)}$$

The introduction of the term in $e^3$ is easily effected, and when M falls near $\pm 90°$, it is better to consider it, in which case it is evident that the terms involving $e^4$ and $e^5$ are very small.

In the first formula the sign of $E-M$ is the same as that of sin M.

In the second formula, $E >$ or $< 180°$, according as $M >$ or $< 180°$. This fact and the sign of cot E are sufficient to determine the quadrant in which E is to be taken.

The following example will sufficiently illustrate the deduction of E by the methods above mentioned:

Suppose $M = 250°$ for the planet *Eurynome* (79), for which $e = [9{\cdot}29078]$.

| Formula C | | Formula D | |
|---|---|---|---|
| $\frac{1}{2}$ | 9·6990 | $e$ | 9·2908 |
| sin M | −9·9730 | Addition log | 0·1962* |
| $e$ | 9·2908 | cos M | −9·5341 |
| cos M | −9·5341 | | 0·2433 |
| $e$ cos M | −8·8249 | cos M − $e$ | −9·7303 |
| | 1·1751 | sin M | −9·9730 |
| $\frac{1}{1+e\cos M}$ | 0·0300* | cot E | 9·7573 |
| $\frac{1}{2}e\sin M$ | −8·9628 | E | 240° 14′ |
| $\sin\frac{1}{2}(E-M)$ | −8·9328 | | |
| $\frac{1}{2}(E-M)$ | −4° 55′ | | |
| E − M | −9 50 | | |
| E | 240 10 | | |

Having obtained an approximate value of E, we shall be greatly assisted in obtaining its true value by considerations such as the following.

In the first series the principal term neglected depends on $\sin M \frac{\sin 3M}{\sin M}$, or $\sin 3M$, which in this case is positive. Hence the deduced value of E is too small.

In the second series the principal term neglected depends on sin M, which in this case is negative. Hence the deduced value of cot E is too great, and since E lies in the third quadrant, it follows that E is too small.

If we then increase E by a few minutes of arc, we can work the results side by side with but little extra labour, the requisite logarithms being taken out at the same entry.

The correct value of E being unknown, the values adopted were 240° 14′ and 240° 20′. With these the calculation is as follows:—

* From Zech's *Tafeln der Additions- und Subtractionslogarithmen.*

| | | |
|---|---|---|
| $e$ | 9·29078 | 9·29078 |
| sin E | −9·93855 | −9·93898 |
| Radius in minutes | 3·53627 | 3·53627 |
| $e'$ sin E | −2·76560 | −2·76603 |
| Natural number | $-582'{\cdot}91 = -9^\circ\ 42'{\cdot}91$ | $-583'{\cdot}49 = -9^\circ\ 43'{\cdot}49$ |
| E | $240^\circ\ 14'$ | $240^\circ\ 20'$ |
| Deduced value of M | 249 56·91 | 250 3·49 |
| $d$M | +3·09 | −3·49 |

We have thus, it will be seen, overstepped the true value of E in our estimate.

We have also

| | | |
|---|---|---|
| $1 - e \cos E$ | 0·04430 | 0·04416 |
| $d$M | +0·48996 | −0·54283 |
| $d$E | $+2'{\cdot}790$ | $-3'{\cdot}153$ |
| E | $240^\circ\ 16'{\cdot}790$ | $240^\circ\ 16'{\cdot}847$ |

The mean of these differs from the true value of E by only $0''{\cdot}31$. Using seven figure logs we find $E = 240^\circ\ 16'\ 49''{\cdot}41$.

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*Note on the Star γ Equulei.* By George Knott, B.A., LL.B.

In the spring of last year Mr. Hind pointed out to me that, in view of the Proper Motion of the star γ *Equulei*, there was ground for the inference that a small 11 mag. companion, detected by me in July 1867, was in physical connection with it. I have been hoping accordingly to obtain further measures of the pair, but have failed doing so until the present month.

I give below the results of the measures hitherto obtained by myself, together with a set by Mr. Burnham at an intermediate epoch:

| | | | |
|---|---|---|---|
| 1867·52 | $P = 276^\circ{\cdot}83$ | $D = 2''{\cdot}173$ | Knott. |
| 1867·57 | 276·85 | 2·089 | —— |
| 1871·61 | 277·62 | 1·875 | —— |
| 1877·72 | 274·5 | 2·1 | Burnham. |
| 1886·83 | 273·76 | 2·058 | Knott. |
| 1886·84 | 272·70 | | —— |

The measures of 1871·61 were taken under unfavourable circumstances, as appears from the note:—" B only to be seen on the most careful scrutiny; obs. most difficult." At my last epoch clouds came over before any measures of distance could be obtained. Mr. Burnham adds another small star, 12 mag., which I have not yet seen. His co-ordinates for it are: $P = 10^\circ{\cdot}0$, $D = 41''{\cdot}3$, Ep. 1877·72. It seems desirable that this star should be re-measured.